

On some Godbillon-Vey classes of a family of regular foliations

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Abstract

The aim of the paper is to construct some Godbillon-Vey classes of a family of regular foliations, defined in the paper. These classes are cohomology classes on the manifold or on suitable open subsets. Some examples are also considered.

Keywords: family of regular foliations, singular foliation, test function, differential form, basic form, cohomology class, Godbillon-Vey class.

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1 Introduction

The families of regular foliations considered in the paper are regular foliations on open subsets such that all the induced leaves on an intersection set give a system of subfoliations as in [2, 7] (i.e. the induced larger-sized leaves are saturated with smaller-sized ones; see conditions (F1)–(F3) in the next section). The resulting geometric distribution, given by the tangent subspaces to leaves of maximal dimension, is a singular one (1. of Proposition 1). Assuming that any intersection is saturated by whole leaves, particular classes of Stefan-Sussmann foliations are obtained (2. of Proposition 1), called here singular foliations that are locally regular.

A tool used to extend Godbillon-Vey forms, on a stratum with a non-minimal dimensional leaves, is the existence of a basic test function on the complement of the stratum. We call a test function, according to a closed subset $M_0 \subset M$, a smooth real function that has M_0 as its set of zeros. The existence of a general test function follows from a classical results of Whitney and some properties of extension of smooth sections on closed subsets (see [8, 11, 16], but in a slight different form). Using the line of [3, Section 4], we give a proof in Proposition 2.

The main constructions in the paper are performed in the fourth section. The most important one is that of the Godbillon-Vey class of leaves of minimal dimension in M and in $\Sigma_{\geq r_i}$ (Theorem 1), where we prove that the Godbillon-Vey form of the leaves extends to a global cohomology class $GV_{\min}(\mathcal{F}) \in H^{1+2q_{\max}}(M)$ (for the leaves of minimal dimension on U_0) and to some Godbillon-Vey classes $GV_{\min}(\mathcal{F}_{\Sigma_{\geq r_i}}) \in H^{2(m-r_i)+1}(\Sigma_{\geq r_i})$ (for the other leaves on U_i , $i > 0$). In the case when there is a basic test function of $M \setminus U_i$, then one get a cohomology class on M (Proposition 5).

Two cases are considered in the last section. First, given a regular foliation \mathcal{F}_0 on M , one can easily construct a family of regular foliations on M (for example, adding in a suitable open set a trivial foliation with one leaf), such that its Godbillon-Vey class $GV_{\min}(\mathcal{F})$ is the same as $GV(\mathcal{F}_0)$, the usual Godbillon-Vey class of \mathcal{F}_0 (Proposition 6). Thus if the Godbillon-Vey class of \mathcal{F}_0 is non-trivial, also is that of the family \mathcal{F} . Second, we prove that if 0 is a regular value for the (weak) test basic function φ_i , then the cohomology class $[\overline{v}_i] \in H^{2q_i+1}(M)$ vanishes (Proposition 7).

Looking at the first example, it seems likely to find a non-trivial family of regular foliations, maybe a singular foliation, that is locally regular, having a more complicated structure and a non-trivial Godbillon-Vey class. The second example shows that a non-trivial Godbillon-Vey class can be found not for a regular (weak) test function, possible for a strong one. We let it as an open problem.

2 Families of regular foliations

Let M be a differentiable manifold. Let us suppose that there is an open cover $\{U_i\}_{i \in I}$ of M such that the following three conditions hold:

(F1) – on every U_i there is a regular foliation \mathcal{F}_i having r_i as dimension of leaves,

(F2) – if $i \neq j$ then $r_i \neq r_j$ and

(F3) – if $U_i \cap U_j \neq \emptyset$, $r_i < r_j$, then $U_i \cap U_j$ is saturated by open subsets of leaves of \mathcal{F}_j and every such open set is saturated to its turn by open subsets of leaves of \mathcal{F}_i .

We can consider a stronger condition than (F3) as:

(F3') – if $U_i \cap U_j \neq \emptyset$, $r_i < r_j$, then $U_i \cap U_j$ is saturated by leaves of \mathcal{F}_j and every such leaf of \mathcal{F}_j is saturated to its turn by leaves of \mathcal{F}_i .

It is easy to see that I is a finite set, $I = \overline{0, k}$. The rank of a point $x \in M$ is $r(x) = \max\{r_i : x \in U_i\}$; if $r(x) = r_i$, then and we denote by \mathcal{D}_x the tangent space to the leaf of \mathcal{F}_i . We denote by $\mathcal{R} = \{r(x) = \dim \mathcal{D}_x : x \in M\}$. If $S \subset M$ then $\mathcal{D}_S = \bigcup_{x \in S} \mathcal{D}_x$ denotes the restriction of \mathcal{D} to S . Let $\mathcal{R} = \{r_i\}_{i=\overline{0, k}}$, where $r_{\min} = r_0 < r_1 < \dots < r_k = r_{\max}$. For $r_i \in \mathcal{R}$, we denote by $\Sigma_{r_i} = \{x \in M : \dim \mathcal{D}_x = r_i\}$, $\Sigma_{< r_i} = \{x \in M : \dim \mathcal{D}_x < r_i\}$, $\Sigma_{\leq r_i} = \{x \in M : \dim \mathcal{D}_x \leq r_i\} = \Sigma_{r_i} \cup \Sigma_{< r_i}$, $\Sigma_{> r_i} = \{x \in M : \dim \mathcal{D}_x > r_i\}$, $\Sigma_{\geq r_i} = \{x \in M : \dim \mathcal{D}_x \geq r_i\} = \Sigma_{r_i} \cup \Sigma_{> r_i}$. We say that the subset $\Sigma_{r_{\min}}$ is the *minimal set* and $\Sigma_{r_{\max}}$ is the *maximal set*. The subsets $\Sigma_{< r_i}$ and $\Sigma_{\leq r_i}$ are closed subsets and their complements, the sets $\Sigma_{\geq r_i}$ and $\Sigma_{> r_i}$ are open closed subsets in M . The subset $\Sigma_{r_i} \subset \Sigma_{\geq r_i}$ is the minimal subset of $\mathcal{D}_{\Sigma_{\geq r_i}}$ and $\Sigma_{> r_i}$ is void if $i = k$ and is equal to $\Sigma_{\geq r_{i+1}}$ if $0 \leq i < k$. We say also that the leaves of \mathcal{F}_i are *leaves of minimal dimension*.

The assignment of a vector subspace $\mathcal{D}_x \subset T_x M$, $(\forall) x \in M$, gives a *singular distribution* \mathcal{D} on M , $\mathcal{D} = \bigcup_{x \in M} \mathcal{D}_x \subset TM$. We denote by $\Gamma_{loc}(\mathcal{D})$ the set of local smooth vector fields tangent to \mathcal{D} in every point where they are defined. One say that \mathcal{D} is:

– *smooth*, if \mathcal{D}_x is spanned by some restrictions to x of some smooth local vector fields from $\Gamma_{loc}(\mathcal{D})$, $(\forall) x \in M$;

– (*completely*) *integrable*, if \mathcal{D} is smooth and there is a partition of M in immersed submanifolds $L \subset M$ such that if $x \in L$, then $\mathcal{D}_x = T_x L$.

(See, for example [1, 15] for more details.)

Proposition 1 .

1. Assuming the conditions (F1), (F2) and (F3), then \mathcal{D} is a smooth singular distribution on M .

2. Assuming the conditions (F1), (F2) and (F3'), then the singular distribution \mathcal{D} is integrable.

Proof. Let $x \in M$ and a regular foliate chart of the leaf F_i of \mathcal{F}_i that contain x , where $r(x) = r_i$. The condition (F3) implies that the canonical tangent vectors to F_i belong to $\Gamma_{loc}(\mathcal{D})$ and their restrictions to x generate $T_x F_i = \mathcal{D}_x$. Assuming supplementary the condition (F3'), then this

local chart is also one corresponding to a singular Stefan-Sussmann foliation on M (according for example to [15]) that is tangent to \mathcal{D} . \square

We say that

- the conditions (F1), (F2) and (F3) define a *family of regular foliations* and
- the conditions (F1), (F2) and (F3') define a *singular foliation that is locally regular*.

For a family of regular foliations, we can define the *leaf* of $x \in M$ as the leaf F_i of \mathcal{F}_i that contains x , of maximal dimension $r(x) = r_i$. Moreover, in general a non-ambiguous leaf can be defined only for totally integrable foliations.

Notice that the conditions (F1), (F2) and (F3) does not always assure that \mathcal{D} (defined as above) is integrable. Indeed, consider the open cover of \mathbb{R}^2 given by $U_1 = \{(x, y) \in \mathbb{R}^2, x > 0\}$ and $U_2 = \{(x, y) \in \mathbb{R}^2, x < 1\}$. Let us consider the foliation \mathcal{F}_1 by one leaf on U_1 and the foliation \mathcal{F}_2 by horizontal lines $y = \text{const.}$ on U_2 . The conditions (F1)-(F3) are fulfilled, but the condition (F3') is not fulfilled. It generates a singular smooth distribution \mathcal{D} that is not integrable, generated by the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \varphi(x) \frac{\partial}{\partial y}$, where φ vanishes for $x \leq 0$ and $\varphi(x) = e^{-\frac{1}{x}}$ for $x > 0$.

Let us consider some other examples.

- Given a family of regular foliations (or a singular foliation that is locally regular), the open set $\Sigma_{\geq r}$ is saturated by leaves of \mathcal{F}_i , where $r_i \geq r$, thus a family of regular foliations (or a singular foliation that is locally regular) $\mathcal{F}_{\geq r}$ is induced. In particular $\mathcal{F}_{\geq r_k} = \mathcal{F}_{r_k}$ on $\Sigma_{\geq r_k} = \Sigma_{r_{\max}}$ is regular.

- A regular foliation on M is a singular foliation that is locally regular, when all the points have the same rank, equal to the dimension of the leaves (i.e. of the foliation).

- A non-trivial example is given by the foliation of \mathbb{R}^n by concentric spheres (as leaves of dimension $n - 1$) and the origin (as a leaf of dimension 0) that is a singular foliation that is locally regular. An other non-trivial example is a singular foliation having as leaves concentric spheres, as in the previous example (of dimension $n - 1$), outside a compact ball $\bar{B}(\bar{0}, \rho) \in \mathbb{R}^n$, $\rho > 0$, while $\bar{B}(\bar{0}, \rho)$ is a union of points (as leaves of dimension 0).

- A singular Stefan-Sussmann foliation on M that has $\mathcal{R} = \{0, r\}$, where $0 < r \leq m = \dim M$ is locally regular. In general, consider a regular foliation \mathcal{F}_U on an open subset $U \subset M$, such that the dimension of leaves is r , where $0 < r \leq m$. The partition of M by the leaves of U and by the points of $\Sigma_0 = M \setminus U$ gives a locally regular Stefan-Sussmann foliation on M . The singular distribution has $\mathcal{R} = \{0, r\}$. Notice that any singular Stefan-Sussmann foliation on M that has $\mathcal{R} = \{0, r\}$ can be obtained in this way.

- Consider a regular foliation \mathcal{F}_U on an open subset $U \subset M$, such that the dimension of leaves is r , where $0 \leq r < m$. Let $\Sigma_0 \subset U$ be a closed subset of M , saturated or not by leaves of \mathcal{F}_U . The partition of M by the leaves of \mathcal{F}_{Σ_0} and the leaf $\Sigma_1 = M \setminus \Sigma_0$ gives a family of regular foliations. This is a singular foliation that is locally regular only if Σ_0 is saturated by the leaves of \mathcal{F}_U , when it gives a locally regular Stefan-Sussmann foliation on M . This singular distribution has $\mathcal{R} = \{r, m\}$.

- Consider some open subsets $U_1, U_2 \subset M$ and a regular foliation \mathcal{F}_1 on U_1 ; we suppose that $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cup U_2 \neq M$. Denote by $\Sigma_0 = M \setminus (U_1 \cup U_2)$ and let $U_0 \supset \Sigma_0$ be an open set. We consider on U_0 and U_2 the trivial foliations \mathcal{F}_0 and \mathcal{F}_2 respectively, where \mathcal{F}_0 has points as leaves and \mathcal{F}_2 has one leaf. It follows a family of regular foliations. If $U_1 \cap U_2$ is saturated by leaves of \mathcal{F}_1 , then the family of regular foliations is a singular foliation that is locally regular.

The suspension constructed for regular foliations (as, for example, in [5, 2.7, 2.8]) can be extended to a family of regular foliations, as follows. Let B and M be two manifolds and \mathcal{F} be

a family of regular foliations or a singular foliation that is locally regular. Let us suppose that $\rho : \pi_1(B) \rightarrow \text{Diff}(M)$ is a representation (i.e. a group morphism) such that every diffeomorphism $\rho(g) \in \text{Diff}(M)$ invariate an open neighborhood U_k of Σ_k , as well as the leaves of the foliation \mathcal{F}_k on U_k that restricts to the leaves on Σ_k . If we denote by \tilde{B} the universal simple connected cover of B , then the suspension space is the quotient space $S = (\tilde{B} \times M)/\sim$ of the equivalence relation $(\tilde{b}, m) \sim (\tilde{b}g, \rho(g)^{-1}m)$, $g \in \pi_1(B)$, on $\tilde{B} \times M$. As in the classical case, one can first consider on $\tilde{B} \times M$ the product foliations \mathcal{F}_0 of the foliation by one leaf on \tilde{B} and the foliations \mathcal{F}_i on M . A family of regular foliations or a singular foliation that is locally regular (accordingly to that on M) is induced on the quotient space S ; the leaves, the sets $\Sigma_{k'}$ of the leaves of a same dimension k' and the open neighborhoods $U'_{k'}$ of $\Sigma_{k'}$ are naturally induced.

As a particular case, consider an open subset $U \subset M$, a regular foliation \mathcal{F}_U on U and $f \in \text{Diff}(M)$ such that $f(U) = U$ and f invariates \mathcal{F}_U . We can consider an open neighborhood W of the closed set $M \setminus U$ (for example $W = M$) and the trivial foliation \mathcal{F}_W by points on W . The leaves of \mathcal{F}_U and the points of $M \setminus U$ as 0-dimensional leaves give a locally regular Stefan-Sussmann foliation on M . The suspension of f is considered for $B = S^1$, $\tilde{B} = \mathbb{R}$, $\pi_1(S^1) = \mathbb{Z}$ and the actions $\mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{R}$, $(x, n) \rightarrow x - n$ and $\mathbb{Z} \times M \rightarrow M$, $(n, m) \rightarrow f^n(m)$.

For example, consider the natural central symmetry $\sigma : S^n \rightarrow S^n \subset \mathbb{R}^{n+1}$, $\sigma(\bar{x}) = -\bar{x}$. Consider also two open spherical caps $C_1 \subset C_2$ centred in the same point A of the sphere S^n and let $C'_1 = \sigma(C_1) \subset C'_2 = \sigma(C_2)$ the symmetric spherical caps centred in $A' = \sigma(A)$, such that $C_2 \cap C'_2 \neq \emptyset$. Denote by $U_1 = S^n \setminus (\bar{C}_1 \cup \bar{C}'_1)$ and by $U_2 = C_2 \cup C'_2$. Consider the trivial foliation \mathcal{F}_2 on U_2 by points and a k -regular foliation \mathcal{F}_1 on U_1 obtained by intersection of U_1 by $k+1$ -parallel planes that can be parallel or not with the support n -hyperplanes of the spherical caps. Obviously the open sets U_1 and U_2 , as well as the foliations \mathcal{F}_1 and \mathcal{F}_2 are invariant by σ . One can consider a quotient locally regular foliation on $\mathbb{R}P^n$, as well as a suspension locally regular foliation on $S = (\mathbb{R} \times S^n)/\sim$, given by the \mathbb{Z} -action $n \cdot (\alpha, \bar{x}) = (\alpha - n, \sigma^n(\bar{x}))$.

3 Test functions

We consider now test functions, that allow us to extend smooth functions and vector fields.

Let $M_0 \subset M$ be a closed subset. We say that a real function $\varphi \in \mathcal{F}(M)$ is a *weak test function* for M_0 if $M_0 = \varphi^{-1}(0)$ (i.e. $\varphi(x) = 0$ iff $x \in M_0$). We say that a weak test function is a *strong test function* for M_0 if, additionally, its values are in $[0, 1]$ and all its differentials vanish in every $x \in M_0$. The existence of test functions is an important tool used in the sequel.

The following simple Lemma shows that the existence of a weak test function gives a strong one.

Lemma 1 *Let $\psi_0 : \mathbb{R} \rightarrow [0, 1]$ be smooth function such that $\psi_0(t) = 0$ iff $t = 0$ and all the derivatives of ψ_0 vanish in $t = 0$. Then for every function $f : M \rightarrow \mathbb{R}$ the function $F = \psi_0 \circ f$ has the same zeros as f and all the differentials of F vanish in its zeros.*

Notice that a function ψ_0 as in Lemma 1 is

$$\psi_0(t) = \begin{cases} \frac{e^{-\frac{1}{t^2}}}{1+e^{-\frac{1}{t^2}}} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \quad (1)$$

A first fact is the existence of a weak test function φ_{M_0} for any closed subset $M_0 \subset M$, i.e. a positive smooth real function on M , having the set of zeros exactly M_0 . The existence follows

from a classical results of Whitney and some properties of extension of smooth sections on closed subsets (see [8, 11, 16]), but in a slight different form. We give a proof below, in line of [3, Section 4].

Proposition 2 *Let M be a differentiable manifold and $M_0 \subset M$ be a closed subset. Then there is a (weak, strong) test function for M_0 .*

Proof. We can proceed as in [3, Section 4] reducing the problem to the case when $M = \mathbb{R}^n$ and considering M properly embedded in \mathbb{R}^k for some k . Then $M_0 \subset \mathbb{R}^k$ is also closed. A test function on \mathbb{R}^k for M_0 reduces to M also to a test function for M_0 . Since M_0 is a closed set, then $M_1 = \mathbb{R}^k \setminus M_0$ is an open subset of \mathbb{R}^k . For any point $p \in M_1$ there is a ball $B_p = B(p, 2r) \subset M_1$. We denote by $B'_p = B(p, r)$ and we consider a bump function $\psi_p : M \rightarrow [0, 1]$ such that its support is $\bar{B}_p = \bar{B}(p, 2r)$, its values are 0 outside B_p (i.e. on $\mathbb{R}^n \setminus B_p$), 1 on $\bar{B}'_p = \bar{B}(p, r)$ and all the other values are in the open interval $(0, 1)$. We can consider an at most countable cover of M_1 with such balls B_p . In the case when the cover of M_1 is a finite set $\{B_i\}_{i=1, \dots, r}$, we can consider $\varphi = \sum_{i=1}^r \psi_i$, that is obviously a test function for M_0 . In the case when the cover of M_1 is a infinite set $\{B_i\}_{i=1, \dots, \infty}$, we can proceed as in [3, Section 4]. For each $i \in \mathbb{N}$ consider the constants c_i such that $c_i \|\psi_i\| \leq 1/2^i$, where the norms are in $BC^\infty(\mathbb{R}^n, \mathbb{R})$, then denote $\varphi_i = c_i \psi_i$ and finally

$$\varphi = \sum_{i=1}^{\infty} \varphi_i.$$

As in the proof of [3, Proposition 4.3], φ is a smooth function and the set of its zeros is $\mathbb{R}^n \setminus M_1 = M_0$, so it is a weak test function for M_0 . Using Lemma 1 with ψ_0 given by the formula (1), we obtain a strong test function for M_0 . \square

The existence of a weak test function that is not a strong one depends on the zero set (i.e. the closed set). For example, the singular foliation of \mathbb{R}^n by concentric spheres (as leaves of dimension $n - 1$) and the origin (as a leaf of dimension 0) is locally regular and the square of the euclidian norm is a weak test function that is not a strong one. The singular foliation having as leaves concentric spheres, as in the previous example (of dimension $n - 1$), outside a compact ball $\bar{B}(\bar{0}, \rho) \subset \mathbb{R}^n$, $\rho > 0$, while $\bar{B}(\bar{0}, \rho)$ is a union of points (as leaves of dimension 0) is also locally regular, but every weak test function of $\bar{B}(\bar{0}, \rho)$ is always a strong one.

4 The construction of Godbillon-Vey forms and classes

Integrability conditions for a regular foliation are given by Frobenius theorem. It can be expressed using differential forms, as, for example, in [14, Ch. 2. and Ch. 3]. We use this in a similar way as in [12]. If a differentiable q -form ν on M has locally the form $\nu = \omega_1 \wedge \dots \wedge \omega_q$, where $\omega_1, \dots, \omega_q$ are local one-forms, we say that ν has rank q .

A regular foliation of co-dimension q on a differentiable manifold M is given by a non-singular global form $\nu \in \Omega^q(M)$ of rank q and, in the locally form $\nu = \omega_1 \wedge \dots \wedge \omega_q$, the local one-forms $\omega_1, \dots, \omega_q$ are sections of the transverse bundle of the foliation, that generate the $\mathcal{F}(M)$ -module of transverse one-forms ([14, Proposition 3.9]). One briefly say that the foliation (or its tangent bundle) is given by $\nu = 0$, or by vanishing of ν .

Let us consider now two regular foliations \mathcal{F}_U and \mathcal{F}_V , $\mathcal{F}_{U|U \cap V} \subset \mathcal{F}_{V|U \cap V}$, such that the tangent bundles of the foliations \mathcal{F}_U and \mathcal{F}_V are given of vanishing the differential forms $\omega_1 \in \Omega^{q_1+q_2}(U)$ and $\omega_2 \in \Omega^{q_2}(V)$ respectively.

Proposition 3 Denoting by $\omega'_1 \in \Omega^{q_1+q_2}(U \cap V)$ and $\omega'_2 \in \Omega^{q_2}(U \cap V)$ the restrictions to $U \cap V$ of ω_1 and ω_2 respectively, where $q_1 > 0$, then there is a differentiable form $\theta \in \Omega^{q_1}(U \cap V)$ such that

$$\omega'_1 = \omega'_2 \wedge \theta. \quad (2)$$

Proof. First, let us suppose that $U = V = U \cap V$ is a domain of coordinates $\{x^u, \tilde{x}^{\bar{u}}, \bar{x}^{\bar{u}}\}$, $u = \overline{1, p}$, $\bar{u} = \overline{1, q_1}$ and $\bar{u} = \overline{1, q_2}$ such that $\{x^u\}$ and $\{x^u, \tilde{x}^{\bar{u}}\}$ are coordinates on the leaves of $\mathcal{F}_{U|U \cap V}$ and $\mathcal{F}_{V|U \cap V}$ respectively. Then $\omega'_1 = h_1 d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^{q_1} \wedge d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^{q_2}$ and $\omega'_2 = h_2 d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^{q_2}$ with $h_1, h_2 \in \mathcal{F}(U \cap V)$ having no zeros, thus relation (2) holds for $\theta = \frac{h_1}{h_2} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^{q_1}$. Returning to the general case, let us consider a partition of unity $\{v_\alpha\}_{\alpha \in A}$ on $U \cap V$ subordinated to a cover with open domain of local foliated charts, as above, where A is finite or $A = \mathbb{N}$. Then define $\theta = \sum_{\alpha \in A} v_\alpha \theta_\alpha \in \Omega^1(U \cap V)$. Since $\omega'_1 = \omega'_2 \wedge \theta_\alpha$ and $\sum_{\alpha \in A} v_\alpha = 1$, then relation (2) holds. \square

In order to avoid coordinates, we consider in the sequel the ideals $\mathcal{I}(\mathcal{F}_U) \subset \Omega^*(U)$ and $\mathcal{I}(\mathcal{F}_V) \subset \Omega^*(V)$ of differential forms that vanish when evaluated with all vectors that are tangent to the leaves of \mathcal{F}_U and \mathcal{F}_V respectively. The two ideals are finitely generated, each homogeneous term containing at least one of the local forms that on $U \cap V$ can be taken of the form $\{\tilde{\omega}^{\bar{u}}, \bar{\omega}^{\bar{u}}\}_{\bar{u}=\overline{1, q_1}, \bar{u}=\overline{1, q_2}}$ and $\{\bar{\omega}^{\bar{u}}\}_{\bar{u}=\overline{1, q_2}}$ respectively. Notice that $d\bar{\omega}^{\bar{u}} = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^{\bar{v}} \wedge \nu_{\bar{v}}^{\bar{u}}$ and $d\tilde{\omega}^{\bar{u}} = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^{\bar{v}} \wedge \nu_{\bar{v}}^{\bar{u}} + \sum_{\bar{v}=1}^{q_1} \tilde{\omega}^{\bar{v}} \wedge \nu_{\bar{v}}^{\bar{u}}$, with $\nu_{\bar{v}}^{\bar{u}}, \nu_{\bar{v}}^{\bar{u}}$ and $\nu_{\bar{v}}^{\bar{u}} \in \Omega^1(U \cap V)$. Then ω_2 has the local form

$$\omega_2 = h_2 \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^{q_2}. \quad (3)$$

The Frobenius theorem used for \mathcal{F}_U and \mathcal{F}_V reads that there are $\mu_1 \in \Omega^1(U)$ and $\mu_2 \in \Omega^1(V)$ such that

$$d\omega_1 = \omega_1 \wedge \mu_1, \quad d\omega_2 = \omega_2 \wedge \mu_2. \quad (4)$$

A product of $q_1 + q_2 + 1$ forms in $\mathcal{I}(\mathcal{F}_U)$ as well as of $q_2 + 1$ forms in $\mathcal{I}(\mathcal{F}_V)$ are null. This enables to consider the closed *Godbillon-Vey forms* $\mu_1 \wedge (d\mu_1)^{q_1+q_2} \in \Omega^{2(q_1+q_2)+1}(U)$ and $\mu_2 \wedge (d\mu_2)^{q_2} \in \Omega^{2q_2+1}(V)$ and the *Godbillon-Vey classes* of the foliations \mathcal{F}_U and \mathcal{F}_V as the cohomology classes $[\mu_1 \wedge (d\mu_1)^{q_1+q_2}] \in H^{2(q_1+q_2)+1}(U)$ and $[\mu_2 \wedge (d\mu_2)^{q_2}] \in H^{2q_2+1}(V)$.

Let us look closely to $U \cap V$, when the relation (2) holds. For sake of simplicity, we use notations ω_1 and ω_2 instead of ω'_1 and ω'_2 respectively.

Differentiating by d (2), then using (4) and the usual properties of the exterior product, we obtain

$$\omega_2 \wedge ((-1)^{q_2} d\theta - \theta \wedge (\mu_1 - (-1)^{q_1} \mu_2)) = 0.$$

Taking into account (3), then

$$d\theta - (-1)^{q_2} \theta \wedge (\mu_1 - (-1)^{q_1} \mu_2) = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^{\bar{v}} \wedge \eta_{\bar{v}}, \quad (5)$$

with $\eta_{\bar{v}} \in \Omega^{q_1}(U \cap V)$. Thus the left side of equality (5) belongs to $\mathcal{I}(\mathcal{F}_V)|_{U \cap V} \subset \Omega^*(U \cap V)$. Denote by

$$\mu_3 = (-1)^{q_2} (\mu_1 - (-1)^{q_1} \mu_2). \quad (6)$$

Differentiating by d and using again the same relation (5), we obtain

$$\theta \wedge d\mu_3 = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^{\bar{v}} \wedge \bar{\eta}_{\bar{v}}, \quad (7)$$

with $\bar{\eta}_{\bar{v}} \in \Omega^{q_1+1}(U \cap V)$, i.e. $\theta \wedge d\mu_3 \in \mathcal{I}(\mathcal{F}_V)|_{U \cap V}$. But using local coordinates as in the proof of Proposition 3, we have that, on a domain U' of such coordinates, there is a local function h_3 such that $\theta - h_3 d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^{q_1} \in \mathcal{I}(\mathcal{F}_V)|_{U'}$. Using this fact in (7), for $U' = U \cap V$, it follows that

$$d\mu_3 = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^{\bar{v}} \wedge \bar{\eta}_{\bar{v}}$$

with $\bar{\eta}_{\bar{v}} \in \Omega^1(U \cap V)$, i.e. $d\mu_3 \in \mathcal{I}(\mathcal{F}_V)|_{U \cap V}$. But $d\mu_2 \in \mathcal{I}(\mathcal{F}_V)|_{U \cap V}$, thus using (6) it follows that $d\mu_1 \in \mathcal{I}(\mathcal{F}_V)|_{U \cap V}$.

Proposition 4 *Assuming $q_1 > 0$, then the following assertions hold true:*

1. *The Godbillon-Vey form and its cohomology class according to the foliation $\mathcal{F}_U|_{U \cap V}$, both vanish.*
2. *If $\mathcal{F}' \subset \mathcal{F}''$, $\mathcal{F}' \neq \mathcal{F}''$, are regular foliations on M and the foliation \mathcal{F}'' has not a null co-dimension, then the Godbillon-Vey class of \mathcal{F}' vanishes.*

Proof. If $q_1 > 0$, then $q_1 + q_2 \geq q_2 + 1$, thus $(d\mu_1)^{q_1+q_2} = 0$ because $(d\mu_1)^{1+q_2} = 0$; it follows that $\mu_1 \wedge (d\mu_1)^{q_1+q_2} = 0$, as well as its cohomology class, thus 1. follows. Then 2. is a simple consequence of 1. \square

The result in Proposition 4 allows to consider the Godbillon-Vey class of the foliation \mathcal{F}_{U_1} having the maximal co-dimension $q_{\max} = m - r_{\min}$, on the open subset $U_{r_{\min}} \subset M$; the foliation has the leaves of minimal dimension. The Godbillon-Vey class is the class $[\mu_{r_{\min}} \wedge (d\mu_{r_{\min}})^{m-r_{\min}}]$. The differential form $GV_{r_{\min}} = \mu_{r_{\min}} \wedge (d\mu_{r_{\min}})^{m-r_{\min}} \in \Omega^{1+2q_{\max}}(U_{r_{\min}})$ is null on any intersection $U_{r_{\min}} \cap U_0 \neq \emptyset$, where U_0 is an open subset corresponding to a foliation \mathcal{F}_{U_0} of co-dimension $q_0 = m - r_0 < q_{\max} = m - r_{\min}$. Thus, extending $GV_{r_{\min}}$ as null outside $U_{r_{\min}}$, we obtain a global closed form that gives $GV_{\min}(\mathcal{F}) \in H^{1+2q_{\max}}(M)$; we call it as the *Godbillon-Vey class on leaves of minimal dimension* of the locally regular foliation \mathcal{F} .

In the general case, let us consider the ascending sequence of open sets $\Sigma_{\geq r_k} \subset \Sigma_{\geq r_{k-1}} \subset \cdots \subset \Sigma_{\geq r_1} \subset \Sigma_{\geq r_0} = M$. Denote by $\mathcal{F}_{\Sigma_{\geq r_i}}$ the restriction of \mathcal{F} to the open set $\Sigma_{\geq r_i}$, $i = \overline{0, k}$; notice that the set $\Sigma_{\geq r_i}$ is saturated by the leaves of $\mathcal{F} = \mathcal{F}_{\Sigma_{\geq r_0}}$. The subset $\Sigma_{r_i} \subset \Sigma_{\geq r_i}$ is that of minimal dimensions of leaves. We can consider the Godbillon-Vey classes $GV_{\min}(\mathcal{F}_{\Sigma_{\geq r_i}}) \in H^{2(m-r_i)+1}(\Sigma_{\geq r_i})$. In particular,

$$GV_{\min}(\mathcal{F}) = GV_{\min}(\mathcal{F}_{\Sigma_{\geq r_0}}) \in H^{2(m-r_0)+1}(\Sigma_{\geq r_0}) = H^{2(m-r_0)+1}(M).$$

Theorem 1 *A Godbillon-Vey form of the leaves extends to a global cohomology class $GV_{\min}(\mathcal{F}) \in H^{1+2q_{\max}}(M)$ (for the leaves of minimal dimension) and to some Godbillon-Vey classes $GV_{\min}(\mathcal{F}_{\Sigma_{\geq r_i}}) \in H^{2(m-r_i)+1}(\Sigma_{\geq r_i})$ (for the leaves on the other U_i , $i > 0$).*

In order to obtain global cohomology classes on M , the construction on the Godbillon-Vey class on the leaves of minimal dimension can be extended to the other strata, provided that there is a foliated test function according to that stratum. We perform below this construction.

Let us suppose that the foliation \mathcal{F}_{r_i} on $U_i \subset M$ has the dimension r_i of leaves and it is defined on U_i by the equation $\omega_i = 0$, where $\omega_i \in \Omega^{q_i}(U_i)$, $q_i = m - r_i$. Then

$$d\omega_i = \omega_i \wedge \mu_i$$

with $\mu_i \in \Omega^1(U_i)$. We suppose below that there is a test function $\varphi_i \in \mathcal{F}(M)$ for $M \setminus U_i$ that restricts to a basic function for the foliation \mathcal{F}_{r_i} on U_i ; we suppose also that $\bar{\mu}_i = \varphi_i \mu_i$ (where μ_i is defined by zero on $M \setminus U_i$) is differentiable on M , i.e. $\bar{\mu}_i \in \Omega^1(M)$; this is always true if φ_i is a strong test function.

Proposition 5 *Let us suppose that the test function φ_i is basic and $\bar{\mu}_i = \varphi_i \mu_i$ is differentiable on M . Then the differential form $\bar{\nu}_i = \bar{\mu}_i \wedge (d\bar{\mu}_i)^{q_i}$ is closed, giving a cohomology class $[\bar{\nu}_i] \in H^{2q_i+1}(M)$.*

Proof. We have $\bar{\nu}_i = \bar{\mu}_i \wedge (d\bar{\mu}_i)^{q_i} = \varphi_i^{1+q_i} \mu_i \wedge (d\mu_i)^{q_i}$. But, if φ_i is basic, then $\psi_i = \varphi_i^{1+q_i}$ is also basic and $d\psi_i \wedge \mu_i \wedge (d\mu_i)^{q_i} = 0$. Thus $d\bar{\nu}_i = 0$ and the conclusion follows. \square

Notice that if the maximal stratum has the dimension $r_k = m$, then its Godbillon-Vey form vanishes, as well as its Godbillon-Vey class. In particular, if a family of regular foliations has $\mathcal{R} = \{r_0, r_1\}$ and $r_1 = m$, then the only possible non-null is the Godbillon-Vey class of the leaves of minimal dimension.

5 Two cases

First, we prove that the usual Godbillon-Vey class of a regular foliation is the same with the Godbillon-Vey class of leaves of minimal dimension of a suitable non-trivial family of regular foliations. Let (M, \mathcal{F}_0) be a regular foliation of co-dimension q_0 defined by a q_0 -differential form $\omega_0 = 0$, such that $d\omega_0 = \omega_0 \wedge \mu_0$. Let us consider two open and non-void subsets W, U_2 having the properties that $\bar{W} \subset U_2$ and $\varphi \in \mathcal{F}(M)$ is a Uryson function such that $\text{supp } \varphi = M \setminus \bar{W} = U_1$. Consider on U_1 the foliation \mathcal{F}_{U_1} as being the restriction to U_1 of foliation \mathcal{F}_0 . Let us suppose that there is on U_2 a non-trivial foliation \mathcal{F}_{U_2} such that its leaves are saturated by leaves of $\mathcal{F}_{0|U_2}$ (for this we can take U_2 the domain of a \mathcal{F}_0 -foliate simple chart and then take as \mathcal{F}_{U_2} a proper foliation having as subfoliation $\mathcal{F}_{0|U_2}$, for example, a trivial foliation with one leaf). The foliation \mathcal{F}_{U_2} is defined by the q_0 -form $\tilde{\omega} = \varphi \omega_0$, that has the same support as φ . The foliations \mathcal{F}_{U_1} and \mathcal{F}_{U_2} give a non-trivial family of regular foliations on M . The Godbillon-Vey class $GV_{\min}(\mathcal{F}) \in H^{2q_0+1}(M)$ is given extending naturally (using Proposition 4) a form that gives the Godbillon-Vey class of \mathcal{F}_{U_1} .

Proposition 6 *The Godbillon-Vey class $GV_{\min}(\mathcal{F})$ is the same as $GV(\mathcal{F}_0)$, the usual Godbillon-Vey class of \mathcal{F}_0 .*

Proof. The Godbillon-Vey class of \mathcal{F}_0 is given by $[\mu_0 \wedge (d\mu_0)^{q_0}]$, where the definition does not depend of ω_0 and μ_0 (see [14, Theorem 3.11]). It can be easily proved that we can take the restriction of ω_0 to U_2 having the form $f d\bar{x}^1 \wedge \cdots \wedge d\bar{x}^{q_0}$, where $\{\bar{x}^{\bar{u}}\}_{\bar{u}=1, q_0}$ are transverse coordinates for \mathcal{F}_0 on U_2 , thus $\mu_0|_{U_2} = (-1)^{q_0} d \log f$ and $d\mu_0|_{U_2} = 0$. Thus the restriction of the differential form $\mu_0 \wedge (d\mu_0)^{q_0}$ to U_2 vanishes and it extends the differential form on U_1 that gives the Godbillon-Vey class of $\mathcal{F}_{0|U_2}$, thus it gives $GV_{\min}(\mathcal{F})$. It follows that $GV_{\min}(\mathcal{F}) = GV(\mathcal{F}_0)$. \square

We consider below a non-trivial case when the Godbillon-Vey class vanishes. More specifically, we prove that for a regular (weak) test function $\varphi_i \in \mathcal{F}(M)$ for $M \setminus U_i$ that restricts to a basic function for the foliation \mathcal{F}_{r_i} on U_i the cohomology class $[\bar{\nu}_i] \in H^{2q_i+1}(M)$ vanishes.

Firstly we shall need some preliminary notions about singular forms and cohomology attached to a function, for more see [9, 10]. Accordingly, for a smooth function $f \in \mathcal{F}(M)$ and $U \subset M$ a p -form $\omega \in \Omega^p(U)$ is called a *singular p -form* if the form $f^p \omega$ can be extended to a smooth form on M , that is $f^p \omega \in \Omega^p(M)$. We denote the space of singular p -forms with respect to f by

$\Omega_f^p(M)$. We notice that if $\omega \in \Omega_f^p(M)$ then $d\omega \in \Omega_f^{p+1}(M)$ and so we have a differential complex $(\Omega_f^\bullet(M), d)$. The cohomology of this differential complex is isomorphic with the cohomology attached to the function f , denoted by $H_f^\bullet(M)$, which is defined as cohomology of the differential complex $(\Omega^\bullet(M), d_f)$, where the coboundary operator $d_f : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ is defined by $d_f\omega = f d\omega - p d f \wedge \omega$. The mentioned isomorphism is produced by the map of chain complexes $\phi : (\Omega_f^\bullet(M), d) \rightarrow (\Omega^\bullet(M), d_f)$ given by $\phi^p : \Omega_f^p(M) \rightarrow \Omega^p(M)$, $\phi(\omega) = f^p \omega$, see [10].

Now, let us return to our study. As well as we seen from the above discussion $\mu_i \in \Omega_{\varphi_i}^1(M)$ and, accordingly $d\mu_i \in \Omega_{\varphi_i}^2(M)$. We have then that $\mu_i \wedge (d\mu_i)^{q_i} \in \Omega_{\varphi_i}^{2q_i+1}(M)$. Since $\mu_i \wedge (d\mu_i)^{q_i}$ is closed, from the above isomorphism we have that $\varphi_i^{2q_i+1} \mu_i \wedge (d\mu_i)^{q_i}$ is d_{φ_i} -closed. Thus, if φ_i is basic function for the foliation \mathcal{F}_{r_i} on U_i then $d_{\varphi_i}(\varphi_i^{q_i} \bar{\nu}_i) = 0$ which leads to the cohomology class $[\varphi_i^{q_i} \bar{\nu}_i] \in H_{\varphi_i}^{2q_i+1}(M)$. Let us consider now the *regular* case for the test function φ_i , that is φ_i does not have singularities in a neighborhood of its zero set (i.e., 0 is a regular value). The subsets $S_i = \varphi_i^{-1}(\{0\}) = M \setminus U_i$ are then embedded submanifolds of M . We also assume that S_i are connected.

We consider some useful notations. Let $V_i \subset V'_i$ be tubular neighborhoods of S_i . We may assume that $V_i = S_i \times]-\varepsilon_i, \varepsilon_i[$ and $V'_i = S_i \times]-\varepsilon'_i, \varepsilon'_i[$, with $\varepsilon'_i > \varepsilon_i$, and that $\varphi_i|_{V'_i} : S_i \times]-\varepsilon'_i, \varepsilon'_i[\rightarrow \mathbb{R}, (x, t) \mapsto t$. We denote by π_i the projections $V'_i \rightarrow S_i$. Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function which is 1 on $[-\varepsilon_i, \varepsilon_i]$ and has support contained in $[-\varepsilon'_i, \varepsilon'_i]$. Note that the function $\rho \circ \varphi_i$ is 1 on V_i , and we can assume that the function $\rho \circ \varphi_i$ vanishes on $M \setminus V'_i$. If ω is a form on S_i , we will denote by $\tilde{\omega}$ the form $\rho(\varphi_i) \pi_i^* \omega$ and notice that $d\varphi_i \wedge d\tilde{\omega} = d\varphi_i \wedge \tilde{d\omega}$, see [10].

According to Theorem 4.1 from [10], if 0 is a regular value of φ_i then, for each $p \geq 1$, there is an isomorphism

$$H_{\varphi_i}^p(M) \cong H^p(M) \oplus H^{p-1}(S_i), \quad (8)$$

given by $\Phi : \Omega^p(M) \oplus \Omega^{p-1}(S_i) \rightarrow \Omega^p(M)$ defined by $\Phi(\alpha, \beta) = \varphi_i^p \alpha + \varphi_i^{p-1} d\varphi_i \wedge \tilde{\beta}$.

Now, taking into account the isomorphism (8) it follows that there exist $\alpha_i \in \Omega^{2q_i+1}(M)$ and $\beta_i \in \Omega^{2q_i}(S_i)$ with $d\alpha_i = d\beta_i = 0$ such that

$$\varphi_i^{q_i} \bar{\nu}_i = \varphi_i^{1+2q_i} \alpha_i + \varphi_i^{2q_i} d\varphi_i \wedge \tilde{\beta}_i. \quad (9)$$

Thus we obtain that $\alpha_i = \varphi_i^{-1-q_i} \bar{\nu}_i - \frac{d\varphi_i}{\varphi_i} \wedge \tilde{\beta}_i$ and by differentiation and taking into account $d\bar{\nu}_i = d\alpha_i = d\beta_i = 0$, one get

$$(-1 - q_i) \varphi_i^{-2-q_i} d\varphi_i \wedge \bar{\nu}_i = 0,$$

where we have used $d\varphi_i \wedge d\tilde{\beta}_i = d\varphi_i \wedge \tilde{d\beta}_i = 0$.

Now, since $d\bar{\nu}_i = 0$ and $d\varphi_i \wedge \bar{\nu}_i = 0$, by Proposition 3.4 from [9] there exist $\bar{\tau}_i \in \Omega^{2q_i-1}(M)$ such that $\bar{\nu}_i = d\varphi_i \wedge d\bar{\tau}_i$ and so $\bar{\nu}_i = d(\varphi_i d\bar{\tau}_i)$. Thus, we obtain the announced result:

Proposition 7 *If 0 is a regular value for the (weak) test function φ_i that is also basic, then the cohomology class $[\bar{\nu}_i] \in H^{2q_i+1}(M)$ vanishes.*

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